Exercise 11

The integration formula

$$\int_0^\infty \frac{dx}{[(x^2-a)^2+1]^2} = \frac{\pi}{8\sqrt{2}A^3}[(2a^2+3)\sqrt{A+a}+a\sqrt{A-a}],$$

where a is any positive number and $A = \sqrt{a^2+1}$, arises in the theory of case-hardening of steel by means of radio-frequency heating.* Follow the steps below to derive it.

(*a*) Point out why the four zeros of the polynomial

$$q(z) = (z^2-a)^2+1$$

are the square roots of the numbers $a \pm i$. Then, using the fact that the numbers

$$z_0 = \frac{1}{\sqrt{2}}(\sqrt{A+a}+i\sqrt{A-a})$$

and $-z_0$ are the square roots of $a+i$ (see Example 3 in Sec. 11), verify that $\pm\overline{z_0}$ are the square roots of $a-i$ and hence that z_0 and $-\overline{z_0}$ are the only zeros of $q(z)$ in the upper half plane $\operatorname{Im} z \geq 0$.

(*b*) Using the method derived in Exercise 8, Sec. 83, and keeping in mind that $z_0^2 = a+i$ for purposes of simplification, show that the point z_0 in part (*a*) is a pole of order 2 of the function $f(z) = 1/[q(z)]^2$ and that the residue B_1 at z_0 can be written

$$B_1 = -\frac{q''(z_0)}{[q'(z_0)]^3} = \frac{a-i(2a^2+3)}{16A^2z_0}.$$

After observing that $q'(-\bar{z}) = -\overline{q'(z)}$ and $q''(-\bar{z}) = \overline{q''(z)}$, use the same method to show that the point $-\overline{z_0}$ in part (*a*) is also a pole of order 2 of the function $f(z)$, with residue

$$B_2 = \overline{\left\{\frac{q''(z_0)}{[q'(z_0)]^3}\right\}} = -\overline{B_1}.$$

Then obtain the expression

$$B_1 + B_2 = \frac{1}{8A^2 i}\operatorname{Im}\left[\frac{-a+i(2a^2+3)}{z_0}\right]$$

for the sum of the residues.

(*c*) Refer to part (*a*) and show that $|q(z)| \geq (R-|z_0|)^4$ if $|z| = R$, where $R > |z_0|$. Then, with the aid of the final result in part (*b*), complete the derivation of the integration formula.

Solution

*See pp. 359–364 of the book by Brown, Hoyler, and Bierwirth that is listed in Appendix 1.

The integrand is an even function of x, so the interval of integration can be extended to $(-\infty, \infty)$ as long as the integral is divided by 2.

$$\int_0^\infty \frac{dx}{[(x^2-a)^2+1]^2} = \int_{-\infty}^\infty \frac{dx}{2[(x^2-a)^2+1]^2}$$

In order to evaluate the integral, consider the corresponding function in the complex plane,

$$f(z) = \frac{1}{2[(z^2-a)^2+1]^2},$$

and the contour in Fig. 99. Singularities occur where the denominator is equal to zero.

$$\begin{aligned} 2[(z^2-a)^2+1]^2 &= 0 \\ (z^2-a)^2+1 &= 0 \\ z^2-a &= \pm i \end{aligned}$$

$$z^2 = a \pm i \quad \to \quad \begin{cases} z_1 = \sqrt{a+i} = \frac{1}{\sqrt{2}}\left(\sqrt{\sqrt{a^2+1}+a} + i\sqrt{\sqrt{a^2+1}-a}\right) \\ z_2 = \sqrt{a-i} = \frac{1}{\sqrt{2}}\left(\sqrt{\sqrt{a^2+1}+a} - i\sqrt{\sqrt{a^2+1}-a}\right) \\ z_3 = -\sqrt{a+i} = \frac{1}{\sqrt{2}}\left(-\sqrt{\sqrt{a^2+1}+a} - i\sqrt{\sqrt{a^2+1}-a}\right) \\ z_4 = -\sqrt{a-i} = \frac{1}{\sqrt{2}}\left(-\sqrt{\sqrt{a^2+1}+a} + i\sqrt{\sqrt{a^2+1}-a}\right) \end{cases}$$

Proof for these alternative representations of z_1, z_2, z_3, and z_4 will be given at the end. The singular points of interest to us are the ones that lie within the closed contour, $z = z_1$ and $z = z_4$.

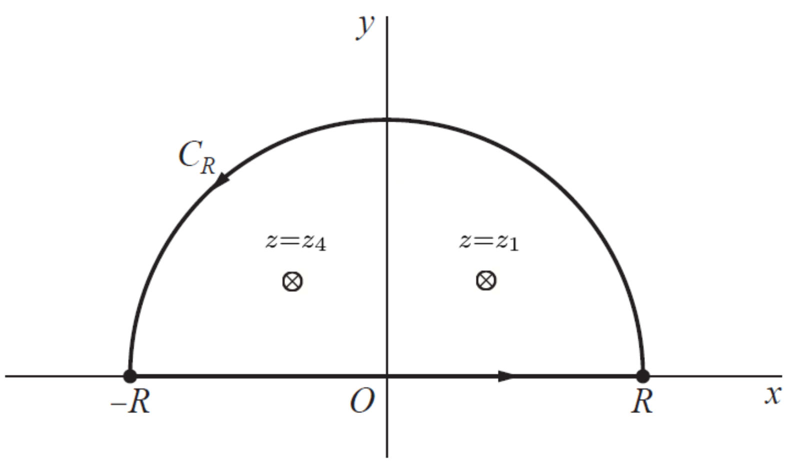

Figure 1: This is Fig. 99 with the singularities at $z = z_1$ and $z = z_4$ marked.

According to Cauchy's residue theorem, the integral of $1/\{2[(z^2-a)^2+1]^2\}$ around the closed contour is equal to $2\pi i$ times the sum of the residues at the enclosed singularities.

$$\oint_C \frac{dz}{2[(z^2-a)^2+1]^2} = 2\pi i \left\{ \operatorname*{Res}_{z=z_1} \frac{1}{2[(z^2-a)^2+1]^2} + \operatorname*{Res}_{z=z_4} \frac{1}{2[(z^2-a)^2+1]^2} \right\}$$

This closed loop integral is the sum of two integrals, one over each arc in the loop.

$$\int_L \frac{dz}{2[(z^2-a)^2+1]^2} + \int_{C_R} \frac{dz}{2[(z^2-a)^2+1]^2} = 2\pi i \left\{ \operatorname*{Res}_{z=z_1} \frac{1}{2[(z^2-a)^2+1]^2} + \operatorname*{Res}_{z=z_4} \frac{1}{2[(z^2-a)^2+1]^2} \right\}$$

The parameterizations for the arcs are as follows.

$$\begin{aligned} L: &\quad z = r, && r=-R \quad \to \quad r=R \\ C_R: &\quad z = Re^{i\theta}, && \theta=0 \quad \to \quad \theta=\pi \end{aligned}$$

As a result,

$$\int_{-R}^{R} \frac{dr}{2[(r^2-a)^2+1]^2} + \int_{C_R} \frac{dz}{2[(z^2-a)^2+1]^2} = 2\pi i \left\{ \operatorname*{Res}_{z=z_1} \frac{1}{2[(z^2-a)^2+1]^2} + \operatorname*{Res}_{z=z_4} \frac{1}{2[(z^2-a)^2+1]^2} \right\}.$$

Take the limit now as $R \to \infty$. The integral over C_R consequently tends to zero. Proof for this statement will be given at the end.

$$\int_{-\infty}^{\infty} \frac{dr}{2[(r^2-a)^2+1]^2} = 2\pi i \left\{ \operatorname*{Res}_{z=z_1} \frac{1}{2[(z^2-a)^2+1]^2} + \operatorname*{Res}_{z=z_4} \frac{1}{2[(z^2-a)^2+1]^2} \right\}.$$

The denominator can be written as $2[(z^2-a)^2+1]^2 = 2[q(z)]^2$. Because $q(z_1)=0$ and $q'(z_1) \neq 0$, $q(z)$ is said to have a zero of order 1 at $z=z_1$ and can be written as

$$q(z) = (z-z_1)\phi_1(z), \tag{1}$$

where $\phi_1(z_1) \neq 0$. Then

$$\begin{aligned} \operatorname*{Res}_{z=z_1} \frac{1}{2[(z^2-a)^2+1]^2} &= \operatorname*{Res}_{z=z_1} \frac{1}{2(z-z_1)^2[\phi_1(z)]^2} \\ &= \operatorname*{Res}_{z=z_1} \frac{1/\{2[\phi_1(z)]^2\}}{(z-z_1)^2}. \end{aligned}$$

We see from this form that $z=z_1$ is a pole of order 2; the residue can then be calculated by

$$\begin{aligned} &= \frac{d}{dz}\left\{\frac{1}{2[\phi_1(z)]^2}\right\}\bigg|_{z=z_1} \\ &= -\frac{\phi_1'(z)}{[\phi_1(z)]^3}\bigg|_{z=z_1} \\ &= -\frac{\phi_1'(z_1)}{[\phi_1(z_1)]^3}. \end{aligned}$$

Differentiate both sides of equation (1) with respect to z twice in order to write the previous formula in terms of $q(z)$.

$$\begin{aligned} q'(z) &= \phi_1(z) + (z-z_1)\phi_1'(z) && \Rightarrow \quad q'(z_1) = \phi_1(z_1) \\ q''(z) &= 2\phi_1'(z) + (z-z_1)\phi_1''(z) && \Rightarrow \quad q''(z_1) = 2\phi_1'(z_1) \quad \to \quad \phi_1'(z_1) = \frac{1}{2}q''(z_1) \end{aligned}$$

Consequently,

$$\begin{aligned}\operatorname*{Res}_{z=z_1} \frac{1}{2[(z^2-a)^2+1]^2} &= -\frac{\frac{1}{2}q''(z_1)}{[q'(z_1)]^3} \\ &= -\frac{1}{2}\frac{[(z^2-a)^2+1]''}{\{[(z^2-a)^2+1]'\}^3}\bigg|_{z=z_1} \\ &= -\frac{1}{2}\frac{4(3z_1^2-a)}{[4z_1(z_1^2-a)]^3}.\end{aligned}$$

Note that $z_1^2 = a+i$.

$$\begin{aligned}&= -\frac{(2a+3i)}{32z_1^3(i)^3} \\ &= -\frac{2a+3i}{32z_1(a+i)(-i)} \\ &= \frac{2a+3i}{32z_1(ia-1)} \cdot \frac{-ia-1}{-ia-1} \\ &= \frac{a-i(2a^2+3)}{32(a^2+1)z_1}\end{aligned}$$

Similarly, because $q(z_4) = 0$ and $q'(z_4) \neq 0$, $q(z)$ is said to have a zero of order 1 at $z = z_4$ and can be written as

$$q(z) = (z-z_4)\phi_2(z), \tag{2}$$

where $\phi_2(z_4) \neq 0$. Then

$$\begin{aligned}\operatorname*{Res}_{z=z_4} \frac{1}{2[(z^2-a)^2+1]^2} &= \operatorname*{Res}_{z=z_4} \frac{1}{2(z-z_4)^2[\phi_2(z)]^2} \\ &= \operatorname*{Res}_{z=z_4} \frac{1/\{2[\phi_2(z)]^2\}}{(z-z_4)^2}.\end{aligned}$$

We see from this form that $z = z_4$ is a pole of order 2; the residue can then be calculated by

$$\begin{aligned}&= \frac{d}{dz}\left\{\frac{1}{2[\phi_2(z)]^2}\right\}\bigg|_{z=z_4} \\ &= -\frac{\phi_2'(z)}{[\phi_2(z)]^3}\bigg|_{z=z_4} \\ &= -\frac{\phi_2'(z_4)}{[\phi_2(z_4)]^3}.\end{aligned}$$

Differentiate both sides of equation (2) with respect to z twice in order to write the previous formula in terms of $q(z)$.

$$\begin{aligned}q'(z) &= \phi_2(z) + (z-z_4)\phi_2'(z) &&\Rightarrow \quad q'(z_4) = \phi_2(z_4) \\ q''(z) &= 2\phi_2'(z) + (z-z_4)\phi_2''(z) &&\Rightarrow \quad q''(z_4) = 2\phi_2'(z_4) \quad \to \quad \phi_2'(z_4) = \frac{1}{2}q''(z_4)\end{aligned}$$

Consequently,

$$\begin{aligned}\operatorname*{Res}_{z=z_4} \frac{1}{2[(z^2-a)^2+1]^2} &= -\frac{\frac{1}{2}q''(z_4)}{[q'(z_4)]^3} \\ &= -\frac{1}{2}\frac{[(z^2-a)^2+1]''}{\{[(z^2-a)^2+1]'\}^3}\bigg|_{z=z_4} \\ &= -\frac{1}{2}\frac{4(3z_4^2-a)}{[4z_4(z_4^2-a)]^3}.\end{aligned}$$

Note that $z_4^2 = a - i$.

$$\begin{aligned} &= -\frac{(2a-3i)}{32z_4^3(-i)^3} \\ &= \frac{2a-3i}{32z_4(a-i)(-i)} \\ &= \frac{2a-3i}{32z_4(-ia-1)}\cdot\frac{ia-1}{ia-1} \\ &= \frac{a+i(2a^2+3)}{32(a^2+1)z_4}\end{aligned}$$

Now that the residues are known, the integral can be evaluated.

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{dr}{2[(r^2-a)^2+1]^2} &= 2\pi i\left\{\operatorname*{Res}_{z=z_1}\frac{1}{2[(z^2-a)^2+1]^2}+\operatorname*{Res}_{z=z_4}\frac{1}{2[(z^2-a)^2+1]^2}\right\} \\ &= 2\pi i\left[\frac{a-i(2a^2+3)}{32(a^2+1)z_1}+\frac{a+i(2a^2+3)}{32(a^2+1)z_4}\right] \\ &= 2\pi i\left[\frac{az_4-iz_4(2a^2+3)+az_1+iz_1(2a^2+3)}{32(a^2+1)z_1z_4}\right] \\ &= 2\pi i\left[\frac{a(z_1+z_4)+i(2a^2+3)(z_1-z_4)}{32(a^2+1)z_1z_4}\right] \\ &= 2\pi i\left[\frac{a\cdot\frac{2i}{\sqrt{2}}\sqrt{\sqrt{a^2+1}-a}+i(2a^2+3)\cdot\frac{2}{\sqrt{2}}\sqrt{\sqrt{a^2+1}+a}}{32(a^2+1)(\sqrt{a+i})(-\sqrt{a-i})}\right] \\ &= 2\pi i\left[(-i)\frac{a\sqrt{\sqrt{a^2+1}-a}+(2a^2+3)\sqrt{\sqrt{a^2+1}+a}}{16\sqrt{2}(a^2+1)^{3/2}}\right] \\ &= \frac{\pi}{8\sqrt{2}(a^2+1)^{3/2}}\left[(2a^2+3)\sqrt{\sqrt{a^2+1}+a}+a\sqrt{\sqrt{a^2+1}-a}\right]\end{aligned}$$

Therefore, changing the dummy integration variable to x,

$$\boxed{\int_0^{\infty}\frac{dx}{[(x^2-a)^2+1]^2} = \frac{\pi}{8\sqrt{2}(a^2+1)^{3/2}}\left[(2a^2+3)\sqrt{\sqrt{a^2+1}+a}+a\sqrt{\sqrt{a^2+1}-a}\right].}$$

The Alternative Representations of z_1, z_2, z_3, and z_4

Start by writing the polar form of $a \pm i$.

$$\begin{aligned}\sqrt{a \pm i} &= \sqrt{\sqrt{a^2+1}\exp\left[i\tan^{-1}\left(\pm\frac{1}{a}\right)\right]} \\ &= \sqrt[4]{a^2+1}\exp\left[\pm\frac{i}{2}\tan^{-1}\left(\frac{1}{a}\right)\right]\end{aligned}$$

Use Euler's formula.

$$= \sqrt[4]{a^2+1}\left\{\cos\left[\frac{1}{2}\tan^{-1}\left(\frac{1}{a}\right)\right] \pm i\sin\left[\frac{1}{2}\tan^{-1}\left(\frac{1}{a}\right)\right]\right\}$$

Apply the half-angle formulas for sine and cosine.

$$= \sqrt[4]{a^2+1}\left\{\sqrt{\frac{1+\cos\left[\tan^{-1}\left(\frac{1}{a}\right)\right]}{2}} \pm i\sqrt{\frac{1-\cos\left[\tan^{-1}\left(\frac{1}{a}\right)\right]}{2}}\right\}$$

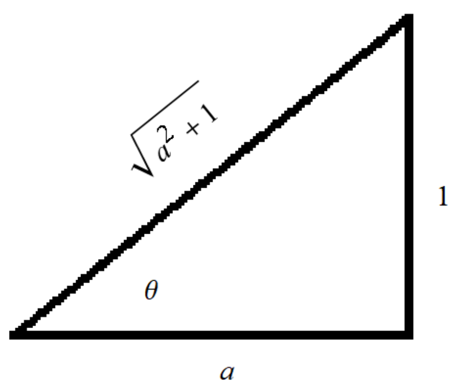

Figure 2: Draw the implied right triangle to determine the cosine of $\theta = \tan^{-1}\left(\frac{1}{a}\right)$.

$$\begin{aligned}\sqrt{a \pm i} &= \frac{\sqrt[4]{a^2+1}}{\sqrt{2}}\left(\sqrt{1+\frac{a}{\sqrt{a^2+1}}} \pm i\sqrt{1-\frac{a}{\sqrt{a^2+1}}}\right) \\ &= \frac{1}{\sqrt{2}}\left(\sqrt{\sqrt{a^2+1}+a} \pm i\sqrt{\sqrt{a^2+1}-a}\right)\end{aligned}$$

Therefore,

$$\begin{aligned}z_1 = \sqrt{a+i} &= \frac{1}{\sqrt{2}}\left(\sqrt{\sqrt{a^2+1}+a} + i\sqrt{\sqrt{a^2+1}-a}\right) \\ z_2 = \sqrt{a-i} &= \frac{1}{\sqrt{2}}\left(\sqrt{\sqrt{a^2+1}+a} - i\sqrt{\sqrt{a^2+1}-a}\right).\end{aligned}$$

The representations for z_3 and z_4 follow because $z_3 = -z_1$ and $z_4 = -z_2$.

The Integral Over C_R

Our aim here is to show that the integral over C_R tends to zero in the limit as $R \to \infty$. The parameterization of the semicircular arc in Fig. 99 is $z = Re^{i\theta}$, where θ goes from 0 to π.

$$\begin{aligned}\int_{C_R} \frac{dz}{2[(z^2-a)^2+1]^2} &= \int_0^\pi \frac{Rie^{i\theta}\,d\theta}{2\{[(Re^{i\theta})^2-a]^2+1\}^2} \\ &= \int_0^\pi \frac{Rie^{i\theta}}{[(R^2e^{i2\theta}-a)^2+1]^2}\,\frac{d\theta}{2}\end{aligned}$$

Now consider the integral's magnitude.

$$\begin{aligned}\left|\int_{C_R} \frac{dz}{2[(z^2-a)^2+1]^2}\right| &= \left|\int_0^\pi \frac{Rie^{i\theta}}{[(R^2e^{i2\theta}-a)^2+1]^2}\,\frac{d\theta}{2}\right| \\ &\le \int_0^\pi \left|\frac{Rie^{i\theta}}{[(R^2e^{i2\theta}-a)^2+1]^2}\right|\frac{d\theta}{2} \\ &= \int_0^\pi \frac{\left|Rie^{i\theta}\right|}{|[(R^2e^{i2\theta}-a)^2+1]^2|}\,\frac{d\theta}{2} \\ &= \int_0^\pi \frac{R}{\left|(R^2e^{i2\theta}-a)^2+1\right|^2}\,\frac{d\theta}{2} \\ &\le \int_0^\pi \frac{R}{[|(R^2e^{i2\theta}-a)^2|-|1|]^2}\,\frac{d\theta}{2} \\ &= \int_0^\pi \frac{R}{(\left|R^2e^{i2\theta}-a\right|^2-1)^2}\,\frac{d\theta}{2} \\ &\le \int_0^\pi \frac{R}{[(|R^2e^{i2\theta}|-|a|)^2-1]^2}\,\frac{d\theta}{2} \\ &= \int_0^\pi \frac{R}{[(R^2-a)^2-1]^2}\,\frac{d\theta}{2} \\ &= \frac{\pi}{2}\frac{R}{[(R^2-a)^2-1]^2}\end{aligned}$$

Now take the limit of both sides as $R \to \infty$.

$$\begin{aligned}\lim_{R\to\infty}\left|\int_{C_R} \frac{dz}{2[(z^2-a)^2+1]^2}\right| &\le \lim_{R\to\infty}\frac{\pi}{2}\frac{R}{[(R^2-a)^2-1]^2} \\ &= \lim_{R\to\infty}\frac{\pi}{2}\frac{R}{\left[R^4\left(1-\frac{a}{R^2}\right)^2-1\right]^2} \\ &= \lim_{R\to\infty}\frac{\pi}{2R^7}\frac{1}{\left[\left(1-\frac{a}{R^2}\right)^2-\frac{1}{R^4}\right]^2}\end{aligned}$$

The limit on the right side is zero.

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{dz}{2[(z^2-a)^2+1]^2}\right| \le 0$$

The magnitude of a number cannot be negative, and only zero has zero magnitude. Therefore,

$$\lim_{R\to\infty}\left|\int_{C_R} \frac{dz}{2[(z^2-a)^2+1]^2}\right| = 0 \quad \to \quad \lim_{R\to\infty}\int_{C_R} \frac{dz}{2[(z^2-a)^2+1]^2} = 0.$$